Indian Statistical Institute
Mid-Semestral Examination
Differential Topology-BMath III

Max Marks: 40 Time: 3 hours

Throughout, $X, Y, Z$ will denote manifolds. All maps are assumed to be smooth.

(1) Decide whether the following statements are true or false. Justify. Answers without correct justifications will not be awarded any marks.
 (a) There does not exist a $1-1$ immersion $f : S^1 \times S^1 \longrightarrow S^2$.
 (b) There exists a $1-1$ immersion $g : S^1 \times S^1 \longrightarrow S^5$.
 (c) If $f : X \longrightarrow Y$ is a diffeomorphism, then $f \pitchfork Z$ for every $Z \subseteq Y$.
 (d) Every smooth map $f : S^2 \longrightarrow S^1$ has a critical point.
 (e) If $X, Z \subseteq Y$ and $X \pitchfork Z$, then
$$T_y(X \cap Z) = T_y(X) \cap T_y(Z)$$
 for every $y \in X \cap Z$. [$3 \times 5 = 15$]

(2) (a) State the pre-image theorem. Show that $O(n)$, the set of all orthogonal matrices is a manifold. Determine its dimension and the tanent space at the identity matrix. [5]
 (b) When do you say that a map is transverse to a manidold. Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Y$ is a sequence of maps. Let $W \subseteq Z$ be a submanifold. Show that $f \pitchfork g^{-1}(W)$ if and only if $(g \circ f)^{-1} \pitchfork W$. [5]

(3) (a) Exhibit a smooth map $f : \mathbb{R} \longrightarrow \mathbb{R}$ whose set of critical values is dense. [8]
 (b) Define the term : *Morse function.* Consider the map $f : S^1 \longrightarrow \mathbb{R}$ defined by $f(x, y) = xy$. Find all critical points of $f$. For any one of the critical points of $f$, decide whether or not it is a non-degenerate critical point. [7]